

Potential theory: the origin and applications

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Abstract

The potential theory comes from mathematical physics, in particular, from electrostatic and gravitational problems and finds their applications in the probability theory, scattering theory, biological systems and many other branches of science. During the last century the classical potential theory and the non-linear potential theory has occupied an important place in mathematics. We present the original gravitational and electrostatic problems that lead to the notion of capacity and harmonic functions. We give the short survey of potential theory and present some modern fields of its applications.

***Keywords:* Potential theory, harmonic functions, stochastic process.**

1. INTRODUCTION AND HISTORY

The term “potential theory” arises from the fact that, in 19th century physics, the fundamental forces of nature were believed to be derived from potentials which satisfied Laplace’s equation. Hence, potential theory was the study of functions that could serve as potentials. Nowadays, we know that nature is more complicated - the equations that describe forces are systems of non-linear partial differential equations, such as the Einstein equations, the Yang-Mills equations, and that the Laplace equation is only valid as a limiting case. Nevertheless, the term “potential theory” has remained as a convenient term for describing the study of functions satisfying the Laplace equation and its generalisations.

As it was mentioned ,originally studies were related to the properties of forces which follow the law of gravitation. In the statement of this law given by I. Newton (1643 - 1727) in 1687 the only forces considered were the forces of mutual attraction acting upon two material particles of small size or two material points. These forces are directly proportional to the product of the masses of these particles and inversely proportional to the square of the distance between them. Thus, the first and the most important problem from the point of view of celestial mechanics and geodesy was to study the

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forces of attraction of a material point by a finite smooth material body - a spheroid and, in particular, an ellipsoid (since many celestial bodies have this shape). After first partial achievements by Newton and others, studies carried out by J. L. Lagrange (1736 - 1813) in 1773, A. Legendre (1752 - 1833) between 1784 - 1794 and by P. S. Laplace (1749 - 1827) continued in 1782 - 1799 became of major importance. Lagrange established that the field of gravitational forces, as it is called now, is potential field. He introduced a function which was called in 1828 by G. Green (1793 - 1841) a *potential function* and later in 1840 by C. F. Gauss (1777 - 1855) - just a *potential*. At present, the achievements of this initial period are included in courses on celestial mechanics.

Already Gauss and his contemporaries discovered that the method of potentials can be applied not only to solve problems in the theory of gravitation but, in general, to solve a wide range of problems in mathematical physics, in particular, in electrostatics and magnetism. In this connection, potential became to be considered not only for the physically realistic problems concerning mutual attraction of positive masses, but also for problems with “masses” of arbitrary sign, or charges. Important boundary value problems were defined, such as the Dirichlet problem and the Neumann problem, the electrostatic problem of the static distribution of charges on conductors or the Robin problem, and the problem of sweeping-out mass (balayage method). To solve the above-mentioned problems in the case of domains with sufficiently smooth boundaries certain types of potentials turned out to be efficient, i. e., special classes of parameter - dependent integrals such as volume potentials of distributed mass, single - and double - layer potentials, logarithmic potentials, Green potentials, etc. Results obtained by A. M. Lyapunov (1857 - 1918) and V. A. Steklov (1864 - 1926) at the end of 19th century were fundamental for the creation of strong methods of solution of the main boundary value problems. Studies in the potential theory concerning properties of different potentials have acquired an independent significance.

In the first half of the 20th century, a great stimulus for the generalisation of the principal problems and the completion of the existing formulations in the potential theory was made on the basis of the general notions of a Radon measure, capacity and generalised functions. Modern potential theory is closely related in its development to the theory of analytic, harmonic and subharmonic functions and to the probability theory.

Together with further studies of classical boundary value problems and inverse problems, the modern period of the development of potential theory is characterised by the applications of methods and notions of topology and functional analysis, and the use of abstract axiomatic methods.

2. SIMPLE EXAMPLES FROM GRAVITY AND ELECTROSTATICS

While the theory of Newtonian potentials has various aspects, it is best introduced as a body of results on the properties of forces which are characterised by *Newton's Law of Universal Gravitation*.

Every particle of matter in the universe attracts every other particle, with a force whose direction is that of the line joining the two, and whose magnitude is directly as the product of their masses, and inversely as the square of their distance from each other.

If we denote by m_1 the mass situated at a point P , by m_2 the mass situated at a point Q , then the force is expressed in the following formula:

$$\vec{F} = \gamma \frac{m_1 m_2}{r^2},$$

where r is the distance between P and Q . The constant of proportionality γ depends solely on the units used. Newton's law is stated in terms of particles. We usually have to deal, not with particles, but with continuously distributed matter. We then naturally think of dividing the body into small parts by the method of the integral calculus, adding the vector forces corresponding to the parts, and passing to the limit as the maximum diameter of the parts approaches 0.

1. Example of straight homogeneous segment. Let us consider a straight line segment, which we regard as having mass, so distributed that the mass on any interval is proportional to the length of the interval. The constant factor of proportionality λ is called the linear density. We have here an idealisation of a straight wire, which is the better approximation the smaller the diameter of the wire relatively to its length and the distance away of the attracted particle.

Let axes be chosen so that the ends of the wire are the points $(0, 0, 0)$ and $(l, 0, 0)$. Let the attracted particle be in line with the wire, at $(x, 0, 0)$, $x > l$, as in Figure 1. Let the wire be divided into intervals by the points $0 = \xi_0, \xi_1, \xi_2, \dots, \xi_n = l$. Then the interval (ξ_k, ξ_{k+1}) carries a mass $\lambda \Delta \xi_k$, which, by our physical law, is to be regarded as concentrated at some point a_k of the interval.

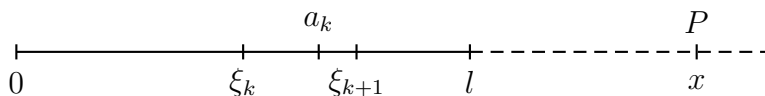


Figure 1

The force, due to the particle thus concentrated, will lie along the x -axis, and will be given, in attraction units, by

$$\Delta X_k = -\frac{\lambda \Delta \xi_k}{|x - a_k|^2}, \quad \Delta Y_k = 0, \quad \Delta Z_k = 0.$$

The force, due to the entire segment $[0, l]$, will be the limit of the sum of the forces due to the system of particles or

$$X = -\int_0^l \frac{\lambda d\xi}{(x - \xi)^2} = -\frac{\lambda l}{x(x - l)}, \quad Y = 0, \quad Z = 0.$$

The result may be given a more suggestive form by introducing the total mass $M = \lambda l$, and by considering where in the segment a particle of that mass should be placed in order to yield the same attraction on a unit particle at $P(x, 0, 0)$. If c is the distance

from this point to P , then

$$X(P) = -\frac{\lambda l}{x(x-l)} = -\frac{M}{c^2} \quad \text{and} \quad c = \sqrt{x(x-l)}.$$

Thus the wire attracts a unit particle at P as if the mass of the wire were concentrated at a point of the wire whose distance from P is the geometric mean of the distances from P to the ends of the wire.

As P approaches the nearest end of the wire, the force becomes infinite, but only like the inverse first power of the distance from this end to P , although a particle would produce a force which becomes infinite like the inverse square of the distance: $X(P) = -\frac{M}{c^2}$. The difference is that in the case of the particle, P moves to the whole mass, whereas in the case of the wire the mass distributed over the segment is concentrated at one of whose points does P moves arbitrary close.

As P recedes further and further away, the *equivalent particle* (as we call the particle with the same mass as the wire, and with the same attraction on a unit particle at P) moves toward the mid-point of the wire, and the attraction of the wire becomes more and more nearly that of a fixed particle at its mid-point. Indeed, if d is the coordinate of the equivalent particle, then

$$d = x - c = x - \sqrt{x(x-l)} \rightarrow \frac{l}{2} \quad \text{as} \quad x \rightarrow \infty.$$

The equivalent particle is situated inside of the wire for any position of P .

2 Example. Let us put the unite mass particle at the point $P(\frac{l}{2})$ on the perpendicular bisector of the material segment as shown in Figure 2

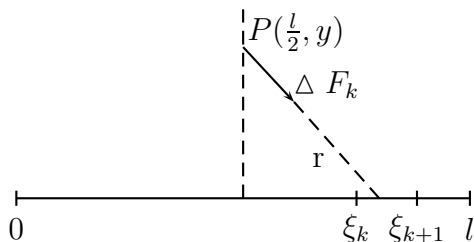


Figure 2

The distance r to the attracted particle from a point $(a_k, 0, 0)$ of the interval (ξ_k, ξ_{k+1}) is given by

$$r^2 = \left(a_k - \frac{l}{2}\right)^2 + y^2,$$

and the magnitude of the force at P , due to a particle at this point, whose mass is that on the interval (ξ_k, ξ_{k+1}) is

$$\Delta F_k = \frac{\lambda \Delta \xi_k}{\left(a_k - \frac{l}{2}\right)^2 + y^2} = \frac{\lambda \Delta \xi_k}{r^2}.$$

Multiplying by the directional cosines $\frac{a_k - \frac{l}{2}}{r}$, $\frac{-y}{r}$, and 0, we get the components of the force as

$$\Delta X_k = \frac{\lambda(a_k - \frac{l}{2}) \Delta \xi_k}{r^3}, \quad \Delta Y_k = \frac{-\lambda y \Delta \xi_k}{r^3}, \quad \Delta Z_k = 0.$$

We integrate from 0 to l . Since the function ΔX_k is even with respect to $a_k - \frac{l}{2}$, the value of integral will be zero. Finally, we get

$$X = 0, \quad Y = -\frac{\lambda l}{y \sqrt{y^2 + \left(\frac{l}{2}\right)^2}} = -\frac{M}{c^2}, \quad Z = 0.$$

The distance between the point P and the equivalent particle is the geometric mean of the distances from P to the nearest and furthest points of the wire. The distance is greater than y . This means that the equivalent particle is situated *beyond* the wire as viewed from P . This fact is significant, as it shows that there does not always exist *in* a body a point at which its mass can be concentrated without altering its attraction for a second body.

3 Example of a material curve. A material curve it is a curve with a prescribed mass density function. Our problem now is to formulate the integrals giving the force exerted by a material curve C on a particle at P . Let the density of C be given as a function λ of the length of arc s of C measured from one end. We assume that λ is continuous. Let C be divided in a usual way into pieces by the points $s_0 = 0, s_1, s_2, \dots, s_n = l$, and let us consider the attraction of a typical piece Δs_k . If λ'_k is the mean value of λ over the piece Δs_k , then the mass of the piece is $\lambda'_k \Delta s_k$. A particle of this mass, situated at a point Q_k of the piece, will exert on a unit particle at $P(x, y, z)$ a force whose magnitude is

$$\Delta F = \frac{\lambda'_k \Delta s_k}{r_k^2}, \quad r_k = |P - Q_k|.$$

If ξ_k, η_k, ζ_k are the coordinates of Q_k , the directional cosines of this force are

$$\cos \alpha = \frac{\xi_k - x}{r_k}, \quad \cos \beta = \frac{\eta_k - y}{r_k}, \quad \cos \gamma = \frac{\zeta_k - z}{r_k},$$

so that the components of the force due to the typical piece are

$$\Delta X_k = \frac{\lambda'_k(\xi_k - x) \Delta s_k}{r_k^3}, \quad \Delta Y_k = \frac{\lambda'_k(\eta_k - y) \Delta s_k}{r_k^3}, \quad \Delta Z_k = \frac{\lambda'_k(\zeta_k - z) \Delta s_k}{r_k^3}.$$

The components in each of the three directions of the axes corresponding to all the pieces of the wire are now to be added, and the limits taken as the lengths of the pieces approach 0. The results will be the components of the force on the unit particle at P

due to the curve

$$(1) \quad \begin{aligned} X &= \int_C \frac{\lambda(s)(\xi(s) - x)}{r^3} ds, \\ Y &= \int_C \frac{\lambda(s)(\eta(s) - y)}{r^3} ds, \\ Z &= \int_C \frac{\lambda(s)(\zeta(s) - z)}{r^3} ds, . \end{aligned}$$

We denote by \vec{r} the vector \overrightarrow{PQ} . Then the formulas (1) can be written as

$$\vec{F} = \int_C \frac{\lambda(s) \vec{r}}{r^3} ds.$$

By analogy, we obtain that the force on the unit particle at P due to a **curved material lamina** S is expressed by the integral

$$\vec{F} = \iint_S \frac{\sigma \vec{r}}{r^3} dS,$$

where σ is the density of the lamina and dS is the element of the area on the curved surface of the lamina. Continuing, we get the force of attracting of a unite mass point P by the ordinary body V

$$\vec{F} = \iiint_V \frac{\kappa \vec{r}}{r^3} dV$$

with the density κ and the volume element dV .

Question. *What is the value of the attracting force if a point P is situated inside the body?*

The integrals (1) are meaningless in this case since the integrands become infinite if the point P is inside the body. But if we remove a small volume v containing P in its interior, the integrands yield definite values. If these values approach limits as the diameter of v approaches 0 these limits are regarded as the components of the force at $P \in V$ due to the whole body.

The next example is from electrostatics.

4 Example of force of charged body. Let e be a positively charged particle located at a point y . If the charged particle with the unit charge is situated at a point x , then the charge e acts on the charge x with the force that is proportional to the product of the charges and inverse to the square of the distance between the points y and x . The force is directed from the point y to the point x . We can write the force as

$$\vec{F}(x) = \frac{e}{r^2} \vec{y}\vec{x} = \frac{e \vec{r}}{r^3},$$

where $\vec{r} = \vec{y}\vec{x}$ and $r = |x - y|$.

Let Ω be a body in \mathbb{R}^3 and we suppose that a positive charge is homogeneously distributed over Ω . Mathematically it means, that there is a continuous function $\rho(x)$

defined in Ω , such that if $B \subset \Omega$ then the part B carried the charge $\int_B \rho dV$, where dV is the volume element. In order to find the force with which the body Ω acts onto the unit charge at a point x , we divide Ω into n small parts B_k , $k = 1, 2, \dots, n$, and choose a point $x_k \in B_k$. The charge of B_k is approximately equal to $\rho(x_k) \Delta V_k$, where ΔV_k is the volume of the part B_k . We think of B_k as a charged particle. Then the total force of this system of n charged particles acting at the point x is

$$\vec{F}_n(x) = \sum_{k=1}^n \frac{\rho(x_k) \Delta V_k}{|x - x_k|^3} (x - x_k).$$

If the diameters of all parts B_k approach 0, then we get

$$\vec{F}(x) = \int_{\Omega} \frac{(x - \xi)}{|x - \xi|^3} \rho(\xi) dV = \int_{\Omega} \frac{\vec{r}}{r^3} \rho dV.$$

The situation in the electrostatics is similar to the example with the gravitational force. It does not matter what kind of physical meaning carries the function $\rho(x)$. The natural mathematical generalisation of the object $\rho(x) dV$ is a measure μ associated to the physical or geometrical setting under consideration. It could be positive measure, stochastic or even vector measure.

3. POTENTIAL

Let us look at the force as on a vector valued function of three variables $F : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$. It is also called the force field. Let e be the charge of a particle located at a point $y \in \mathbb{R}^3$. If it is positive, it also can be thought of as a mass of the particle. As we saw in the previous Section the force that the particle e produces at a point $x \in \mathbb{R}^3$ is

$$(2) \quad F(x) = \frac{e}{|x - y|^3} (x - y).$$

The force has a singularity at $x = y$. Moreover, it is the *potential* field. It means, that there is a function $\varphi : \Omega \rightarrow \mathbb{R}$, such that $F(x) = \nabla \varphi$. The function φ is called the *potential*. We have $\varphi = -\frac{e}{|x-y|}$ for the force field (2).

The force field created by a system of n charged particles e_1, e_2, \dots, e_n situated at points y_1, y_2, \dots, y_n is equal to

$$F(x) = \sum_{k=1}^n \frac{e_k}{|x - y_k|^3} (x - y_k).$$

It is also the potential field with the potential $\varphi = -\sum_{k=1}^n \frac{e_k}{|x - y_k|}$. In general, any force field of attraction generated by a charged body Ω (material curve or lamina) is given by the formula

$$F(x) = \int_{\Omega} \frac{\rho(y)}{|x - y|^3} (x - y) dV$$

and it is the potential field with the potential

$$\varphi(x) = - \int_{\Omega} \frac{\rho(y) dV}{|x - y|}.$$

Here $\rho(y)$ is a density function of the distribution of the charge over the body Ω (material curve or lamina) and dV is the volume element (length element on the curve, area element on the surface). So studying the properties of the potential, we take the information about the attracting force and the distribution of this force over the space and over the body.

We can generalise the notion of the potential, changing the mass element $\rho(y) dV$ by a more general notion of the Borel measure $\mu(y)$. Let us denote by $\mathcal{M}^+(\mathbb{R}^3)$ the set of all positive Borel measures on \mathbb{R}^3 with the finite total mass $\int_{\mathbb{R}^3} d\mu(y) < \infty$.

Definition 1. The (Newton) potential of a measure $\mu \in \mathcal{M}^+(\mathbb{R}^3)$ is the function

$$U^\mu(x) = \int_{\mathbb{R}^3} \frac{d\mu(y)}{|x - y|}.$$

The n -dimensional, $n > 3$, generalisation of Definition 1 is the following.

Definition 2. The (Newton) potential of a measure $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ is the function

$$U^\mu(x) = \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x - y|^{n-2}}.$$

Question. If μ is a measure and we know its potential U^μ , then can we say something about the measure? It is a quite practical question. If we can measure the potential that produces a tumour in the human body or a well of iron at each point outside, what can we know about the size of the tumour or the well and the distribution of the mass inside of them?

This problem is related with the physical properties of the force field. Let us consider this problem by the example of the electric force field. The electric field (2) is the *divergence free*, or *solenoidal*, it means that

$$\operatorname{div} F(x) = \frac{\partial F}{\partial x_1} + \frac{\partial F}{\partial x_2} + \frac{\partial F}{\partial x_3} = 0 \quad \text{for } x \neq y.$$

This immediately implies

Proposition 0.1. If Σ is a closed surface surrounding the charge e , then the flux of the electric field (2) through Σ is equal to $4\pi e$:

$$\int_{\Sigma} \vec{F} \cdot \vec{n} \, dS = 4\pi e.$$

Proof. We take a small ball B with the centre at the point where the charge e is situated. Then $F(x)$ is smooth in the domain G between the surface Σ and the boundary of B and we can apply the Stokes theorem. Since the field is divergence free in G , we change the integration over the Σ by the integration over the boundary of B and finish the proof. \square

If the charged particle is not inside of the closed surface, the flux will vanish. If the surface encloses the system of n charged particles e_1, \dots, e_n , the flux is expressed by the formula

$$\int_{\Sigma} \vec{F} \cdot \vec{n} \, dS = 4\pi \sum_{k=1}^n e_k.$$

Continuing, we consider the surface Σ that is the boundary of a charged body Ω with the density of the charge $\rho(y)$ distributed over Ω . Then the flux of the electric field through Σ is

$$(3) \quad \int_{\Sigma} \vec{F} \cdot \vec{n} \, dS = 4\pi \int_{\Omega} \rho \, dV.$$

Assuming that the field $F(x)$ is smooth, we apply the Stokes theorem and get

$$\int_{\Omega} \operatorname{div} F \, dV = 4\pi \int_{\Omega} \rho \, dV.$$

If we reduce Ω to a point $x_0 \in \Omega$, then we get $\operatorname{div} F(x_0) = 4\pi\rho(x_0)$. If a point y does not belong to Ω , then $\operatorname{div} F(y) = 0$. Remember that $F = \nabla\varphi$, $\varphi(x) = -\int_{\Omega} \frac{\rho(y) \, dV}{|x-y|}$, and that $\operatorname{div}(\nabla)\varphi = \Delta\varphi$, where $\Delta\varphi = \sum_{k=1}^3 \frac{\partial^2\varphi}{\partial x_k^2}$. Applying this observation to (3), we conclude that

$$-\Delta\varphi(x) = 2\pi\rho(x), \quad x \in \Omega,$$

$$(4) \quad \Delta\varphi(x) = 0, \quad x \notin \Omega.$$

The function φ that has the second order continuous partial derivatives and satisfies the equation (4) is called *harmonic*. Equation (4) is called the Laplace equation. It leads to the hypothesis that the potential is a harmonic function and the properties of the potential are closely related with the properties of harmonic functions. In those subjects where a physical law requires the Laplace equation we can find applications of the potential theory. Since the real and imaginary parts of an analytic function of one complex variable are harmonic, the potential theory on the complex plane is essentially the theory of analytic functions. In this case we can use the powerful technique developed for analytic functions in order to study potentials.

4. CAPACITY

We start from another physical problem that leads to deep generalisations. Let Ω be a conducting body and let Σ be a conducting closed surface enclosing the body Ω and connected with earth. Let charge the body with the charge e and see what happens on Σ . The charge e will be distributed over Ω and will induce charges on Σ until an equilibrium is achieved. The charged body produces an electric field F in \mathbb{R}^3 with a potential φ . The field F vanishes inside of the body Ω . An equilibrium distribution of the charge will be established between the surfaces $\partial\Omega$ and Σ . Let us denote by ω_1 and ω_2 the densities of distributed charges over $\partial\Omega$ and Σ correspondingly. We suppose that the densities are continuous functions over the surfaces. The potential

$$\varphi(x) = \int_{\partial\Omega} \frac{\omega_1(y) \, dS}{|x-y|} + \int_{\Sigma} \frac{\omega_2(y) \, dS}{|x-y|}$$

is a continuous function in \mathbb{R}^3 . Moreover, $\varphi = 0$ on Σ since the surface is grounded. We have also $F = -\nabla\varphi = 0$ inside of the body Ω that yields that φ is constant inside Ω . Since the function φ is continuous, there is a constant V such that $\varphi = V$ on $\partial\Omega$. The

value φ is also called the difference of potentials between $\partial\Omega$ and Σ . The value of V is proportional to the value of the charge e , and the coefficient of the proportion is called the *capacity*. Thus $C = \frac{e}{V}$ is the capacity of the condenser formed by $\partial\Omega$ and Σ .

Let us fix $\partial\Omega$ and the charge e . Let turn the surface Σ to the infinity. We get the potential φ that possesses the following properties for each position of the surface Σ :

1. the function φ is harmonic between $\partial\Omega$ and Σ ,
2. the function φ is constant on $\partial\Omega$,
3. $\varphi(x) = 0$ for $x \in \Sigma$.

This family of functions tends to a limit function φ_∞ with the following characteristics:

1. the function φ_∞ is harmonic between $\partial\Omega$ and ∞ ,
2. the function φ is constant on $\partial\Omega$,
3. $\varphi_\infty = 0$ at ∞ .

We expect that a distribution μ_∞ of the charge e on Ω will correspond to the limit function φ_∞ . So

$$\varphi_\infty(x) = \int_{\partial\Omega} \frac{d\mu_\infty(y)}{|x-y|}, \quad \int_{\partial\Omega} d\mu_\infty(y) = e.$$

Distribution (measure) μ_∞ is called *equilibrium distribution (measure)* of the charge e and the function φ_∞ is the *equilibrium potential*. The value of φ_∞ at x is equal to the work applied to transfer the unit charge from the point x to infinity.

For each charge e distributed over Ω there corresponds the equilibrium potential that admits a constant value V_∞ on Ω . The constant $C_\infty = e/V_\infty$ is called the capacity of the body Ω . We conclude that the capacity of a body it is the value of the charge on Ω that produces the unit difference of the potentials between $\partial\Omega$ and ∞ under the equilibrium distribution of charge.

Any distribution of charge on Ω possesses potential energy and it is natural to expect that the equilibrium state corresponds to the minimum of potential energy that has a system of the distributed charges. What is the mathematical expression for the potential energy of the system of charges e_1, \dots, e_n situated at points x_1, \dots, x_n ? When we put the charges e_1, \dots, e_n into the points x_1, \dots, x_n the charges start to move and after some time the system comes to the equilibrium state carrying out some work. The total work that the system produced is called the *potential energy of the system*. To calculate the total potential energy, we fix first $n-1$ charged particles e_1, \dots, e_{n-1} and leave the particle e_n free. The particle e_n will be moved to the infinity by the force $F = -e_n \nabla \varphi$, $\varphi(x) = \sum_{k=1}^{n-1} \frac{e_k}{|x-x_k|}$. Let $x = x(s)$, $s \in [x_n, \infty)$, be a trajectory of e_n with $x(0) = x_n$, $x(\infty) = \infty$, parametrised by the arc length parameter s . The total work W_n is

$$W_n = -e_n \int_0^\infty \nabla \varphi \cdot \frac{dx}{ds} ds = -e_n \int_0^\infty \frac{d\varphi}{ds} ds = e_n \varphi(x_n) = \sum_{k=1}^{n-1} \frac{e_k e_n}{|x_n - x_k|}.$$

Making the charge e_{n-1} free, we get

$$W_{n-1} = \sum_{k=1}^{n-2} \frac{e_k e_{n-1}}{|x_{n-1} - x_k|},$$

analogously. The total work, when all charges are free, is the sum

$$(5) \quad W = \sum_{k=1}^n W_k = \sum_{k < l} \frac{e_k e_l}{|x_k - x_l|} \quad \text{or} \quad W = \frac{1}{2} \sum_{k \neq l} \frac{e_k e_l}{|x_k - x_l|}.$$

There is a natural way to generalise the potential energy (5) to the situation of the potential energy produced by a charged body $\Omega \in \mathbb{R}^3$. Let $\mu \in \mathcal{M}^+(\mathbb{R}^3)$. We define the energy of the measure μ by the equality

$$\mathcal{E}(\mu) = \iint \frac{d\mu(x)d\mu(y)}{|x - y|}.$$

We expect that the equilibrium distribution ν of the charge e on the body Ω will minimise the energy integral. Namely,

$$\mathcal{E}(\nu) \leq \mathcal{E}(\mu), \quad \forall \mu \in \mathcal{M}^+(\mathbb{R}^3) \quad \text{with} \quad \int_{\Omega} d\mu = e.$$

Further generalisation gives

$$\mathcal{E}(\mu) = \iint \frac{d\mu(x)d\mu(y)}{|x - y|^{n-2}}$$

for $n \geq 3$. Involving Definition 2 of the potential U^μ , we write

$$\mathcal{E}(\mu) = \int U^\mu(x) d\mu(x).$$

Let us normalise the equilibrium measure, setting $\nu(\Omega) = C(\Omega)$. Then

$$U^\nu(x) = 1 \quad \text{for} \quad x \in \Omega$$

and

$$\mathcal{E}(\nu) = \int_{\Omega} 1 d\nu(x) = \nu(\Omega) = C(\Omega).$$

We are ready to give the definition of a new mathematical notion *capacity* as the following variational problem:

$$C(\Omega) = \inf \{ \mathcal{E}(\mu) : \mu \in \mathcal{M}^+(\Omega), U^\mu \geq 1 \text{ in } \Omega \}.$$

Notice

$$\int_{\mathbb{R}^3} |\nabla U^\mu|^2 dx = \int_{\mathbb{R}^3} \nabla U^\mu \cdot \nabla U^\mu dx = - \int_{\mathbb{R}^3} U^\mu \cdot \Delta U^\mu dx = 4\pi \mathcal{E}$$

since $\Delta U^\mu = 4\pi\mu(x)$ on Ω and $\Delta U^\mu = 0$ on $\mathbb{R}^3 \setminus \Omega$. If we denote the potential $U^\mu(x)$ by $u(x)$ and observe that the function $u(x)$ has a compact support, then we come to the following definition.

Definition 3. The following value

$$C(\Omega) = \inf \left\{ \frac{1}{4\pi} \int_{\mathbb{R}^3} |\nabla u|^2 dx : u \in C_0^\infty(\mathbb{R}^3), u \geq 1 \text{ in } \Omega \right\}$$

is called the capacity of the domain $\Omega \subset \mathbb{R}^3$.

The extremal function φ coincides with the equilibrium potential U^ν of the normalised equilibrium measure ν .

The concept of capacity is indispensable to an understanding of the local behaviour of functions in a Sobolev space. Capacity takes the place of measure in Egorov and Lusin type theorems for Sobolev functions. The capacity estimates play a decisive role in studies of solutions to partial differential equations. The following developing of the capacity theory finds itself in a varying of the power of $|\nabla u|$. The definition (3) gives the definition of so-called *conformal capacity*, closely related with the theory of functions of one complex variable and the Hardy space H_2^1 . We define now the capacity related to more general functional spaces like the Sobolev space W_p^α , where $p > 0$ is the summability index and α is the differentiability index, that, in general, can be any real number. If we assume that $\alpha \in \mathbb{N}$, then the Sobolev space is a space of functions having weak derivatives of order α , that are summable with the power p . The norm in the Sobolev space W_p^α is defined as

$$\|u\|_{W_p^\alpha}^p = \int_{\Omega} |\nabla^\alpha u|^p < \infty,$$

where $\nabla^\alpha u$ is the weak derivative of order α of the function $u \in \Omega \subset \mathbb{R}^n$.

Definition 4. Let K be a compact in \mathbb{R}^n . Then

$$(6) \quad C_{\alpha,p}(K) = \inf \{ \|u\|_{W_p^\alpha}^p : u \in C_0^\infty(\mathbb{R}^n), u \geq 1 \text{ on } K \}$$

called the capacity of the compact K .

If $\alpha = 1$ and $p = 2$, then Definition 4 is reduced to Definition 3 of the conformal capacity. The Euler-Lagrange equation for the variational problem stated for u in Definition 4 are non-linear and extremely hard. For example, we get the equation

$$\operatorname{div}(\nabla u |\nabla u|^{p-2}) = 0$$

for $\alpha = 1$ and $p > 0$ which is called the p -Laplace equation. The name comes from the particular case $p = 2$, because we get the Laplace equation $\operatorname{div}(\nabla u) = \Delta u = 0$. The function u for arbitrary α and p is still called the potential. The theory based on studying properties of the function u belongs to the non-linear potential theory related to general non-linear equations

$$(7) \quad \operatorname{div} \mathcal{L}(x, u, \nabla u) = 0.$$

The mapping $\mathcal{L}(x, \xi, \eta)' : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies the following conditions for some constants $0 < \alpha \leq \beta < \infty$:

1. the mapping $x \mapsto \mathcal{L}(x, u, \nabla u)$ is measurable for all $\xi, \eta \in \mathbb{R}^n$ and the mapping $(\xi, \eta) \mapsto \mathcal{L}(x, \xi, \eta)$ is continuous for almost every $\xi, \eta \in \mathbb{R}^n$,
- and for all $\xi, \eta \in \mathbb{R}^n$ and for almost every $\xi, \eta \in \mathbb{R}^n$
2. *ellipticity:* $\mathcal{L}(x, \xi, \eta) \cdot \eta \geq \alpha |\eta|^p$,
3. *continuity:* $|\mathcal{L}(x, \xi, \eta)| \leq \beta |\eta|^{p-1}$,
4. *monotonicity:* $(\mathcal{L}(x, \xi, \eta_1) - \mathcal{L}(x, \xi, \eta_2)) \cdot (\eta_1 - \eta_2) > 0$ for any $\eta_1, \eta_2 \in \mathbb{R}^n$, $\eta_1 \neq \eta_2$,
5. *homogeneity:* $\mathcal{L}(x, \xi, \lambda \eta) = \lambda |\lambda|^{p-2} \mathcal{L}(x, \xi, \eta)$ whenever $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

Roughly speaking, it is supposed that $\mathcal{L}(x, \xi, \eta)$ has a similar to the p -Laplacian growth, namely:

$$\mathcal{L}(x, \xi, \eta) \cdot \eta \approx |\eta|^p.$$

5. FURTHER DEVELOPMENT OF POTENTIAL THEORY

In the general case, the solutions to the Euler-Lagrange equation (7) for the variational problem (6) form a cone: if u is a solution and $\lambda, \tau \in \mathbb{R}$, $\lambda > 0$, then $\lambda u + \tau$ is a solution to (7). The continuous solutions to (7) are still often called harmonic functions. Only for $\alpha = 1$, $p = 2$ the space of solutions is a linear vector space. Nevertheless, the *comparison principle*: if u, v are harmonic and $u \leq v$ on the boundary of a bounded domain Ω , then $u \leq v$ inside Ω , makes it possible to develop a genuine potential theory associated with the solutions to (7) without having a linear solution space. The supremum of any harmonic function is controlled by its infimum on compact sets. The corresponding inequality

$$\sup u \leq K \inf u$$

is called the Harnack inequality. The comparison principle is the basis for definition of superharmonic and subharmonic functions. A super(sub)harmonic function is defined as a lower(upper)continuous function u which is not identically infinite and for each solution to (7) the comparison principle holds on any open set. More precisely: for each open set D and each continuous up to the boundary ∂D of D harmonic function h the inequality

$$u \geq h \text{ on } \partial D \text{ implies } u \geq h \text{ in } D.$$

The monotone limit of a sequence of super(sub)harmonic functions locally converges to a super(sub)harmonic function. An intriguing fact is that the definition, which requires no a priori regularity, leads to the existence and the integrability of the gradient of a locally bounded super(sub)harmonic function. This enables us to use variational methods in potential theoretic problems and vice versa. The balayage method, Perron method, Dirichlet principle are used to solve the Dirichlet problem for equations of type (7) in domains with regular boundary. Polar sets, harmonic measures, fine topologies, are powerful tools in studying the boundary behaviour of harmonic functions.

The studies of the potentials with other kernels, different from the Newtonian $\frac{1}{|x-y|^{n-2}}$, like Riesz, Bessel, logarithmic, Green potentials and the study of the Dirichlet problem for compacts and the stability of the Dirichlet problem are of great importance in the applications to the solution of boundary value problems.

6. STOCHASTIC PROCESSES AND POTENTIAL THEORY

The last part of these notes we would like to devote to the relation between the potential theory and the stochastic processes theory which can be considered as a further generalisation of the potential theory. Now the modern potential theory now studies stochastic processes with their adjacent theory, higher order differential operators and

their combination like stochastic differential equations. The behaviour of many systems, like sensors monitoring heart beats, management of the risks involved in using stocks to fund pensions, can be profoundly affected by stochastic fluctuations and randomness. Significant benefits can be achieved if one can be quantitative, and where possible deeply understand, these systems. For example, the ability to price risk, has radically changed the financial markets, and is at the present time causing a complete rearrangement of the conventional insurance industry. The mathematical study of such systems will almost certainly involve tools with such names as martingales, paths space, Malliavin calculus, stochastic integrals, measure valued branching processes, stochastic partial differential equations etc. All of them are central tools for stochastic analysis. The mathematical analysis of stochastic systems, while perhaps a little inaccessible to outsiders, is undergoing quite rapid scientific development.

We start from the clarification of what is a stochastic process. In mathematics of probability, a stochastic process or a random process is a process that can be described by a probability distribution. The two most common types of stochastic processes are the time series, which has a time interval domain, and the random fields, which has a domain over a region of space.

Familiar examples of processes modeled as stochastic time series include stock market and exchange rate fluctuations, signals such as speech, audio and video - medical data such as a patient's EKG, EEG, blood pressure or temperature; and random movement such as Brownian motion or random walks. Examples of random fields include static images, random terrain (landscapes), or composition variations of an inhomogeneous material.

We fix on two examples: martingales and Markov processes. A martingale is a stochastic process (i. e., a sequence of random variables) such that a conditional expected value of an observation at some time t , given all the observations up to some earlier time s , is equal to the observation at that earlier time s . Originally, martingale referred to a class of betting strategies in 18th century France. The simplest of these strategies was designed for a game in which the gambler wins his stake if a coin comes up heads and loses it if the coin comes up tails. The strategy had the gambler double his bet after every loss, so that the first win would recover all previous losses plus win a profit equal to the original stake. Since as a gambler's wealth and available time jointly approach infinity his probability of eventually flipping heads approaches 1, the martingale betting strategy was seen as a sure thing by those who practised it. Of course, in reality the exponential growth of the bets would eventually bankrupt those foolish enough to use the martingale for a long time.

A Markov process is a stochastic process satisfying the Markov property: the conditional probability distribution of future states of the process, given the present state and all past states, depends only upon the present state and not on any past states, i. e. it is conditionally independent of the past states (the path of the process) given the present state. It distinguishes the Markov chain and a continuous - time Markov process. A Markov chain is a series of states of a system that at each time may have changed from the state it was in the moment before, or it may have stayed in the same state (Markov property). The changes of state are called transitions. A continuous - time Markov process is a stochastic process $\{X(t) : t \geq 0\}$ that satisfies the Markov

property and takes values from amongst the elements of a discrete set called the state space. The Markov property in this case can be stated as follows: at any time $t > s > 0$ the conditional probability distribution of the process at time t given the whole history of the process up to and including time s , depends only on the state of the process at time s . In effect, the state of the process at time t is conditionally independent of the history of the process before time s , given the state of the process at time s .

Potential theory and certain aspects of probability theory are intimately related, perhaps most obviously in that the transition function determining a Markov process can be used to define the Green function of a potential theory. Thus it is possible to define and develop many potential theoretic concepts probabilistically, a procedure potential theorists observe with jaundiced eyes in view of the fact that now as in the past their subject provides the motivation for much of Markov process theory. However that may be it is clear that certain concepts in potential theory correspond closely to concepts in probability theory, specifically to concepts in martingale theory. For example, superharmonic functions correspond to supermartingales. More specifically: the Fatou type boundary limit theorems in potential theory correspond to supermartingale convergence theorems; the limit properties of monotone sequences of superharmonic functions correspond surprisingly closely to limit properties of monotone sequences of supermartingales. Certain positive superharmonic functions (supermartingales) are called “potentials”, have associated measures in their respective theories and are subjects to domination principles (inequalities) involving the supports of those measures; in each theory there is a reduction operation whose properties are the same in the two theories and these reductions induce sweeping (balayage) of the measure associated with potentials, and so on.

There is a natural criticism saying that there is no reason to treat the very special potential theories of the Laplace and heat equations rather than general axiomatic potential theory. Another criticism is that there is no reason to treat potential theory other than a special subhead of Markov process theory. However, classical potential theory is too important to serve merely as a source of illustrations for axiomatic potential theory, which theory in turn is too important in its own right to be left to probabilists. To learn potential theory from probability is like to learn algebraic geometry without geometry.

5. Example of weather model. At the end we present an example of a very simple weather model that illustrates the Markov chain. The probabilities of weather conditions, given the weather on the preceding day, can be represented by a transition matrix:

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}.$$

The matrix P represents the weather model in which a sunny day is 90% likely to be followed by another sunny day, and a rainy day is 50% likely to be followed by another rainy day. The columns can be labelled “sunny” and “rainy” respectively, and the rows can be labelled in the same order. P_{ij} is the probability that, if a given day is of type i , it will be followed by a day of type j . Notice that the rows of P sum to 1: this is because P is a stochastic matrix. How to predict the weather in this model? The weather on day 0 is known to be sunny. This is represented by a vector in which the “sunny” entry is 100%, and the “rainy” entry 0%: $x^{(0)} = (1, 0)$. The weather on day 1 can be predicted

by:

$$x^{(1)} = x^{(0)}P = (1, 0) \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = (0.9, 0.1).$$

Thus, there is an 90% chance that day 1 will sunny. The weather on day 2 can be predicted in the same way.

$$x^{(2)} = x^{(1)}P = x^{(0)}P^2 = (1, 0) \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}^2 = (0.86, 0.14).$$

General rules for day n are

$$x^{(n)} = x^{(n-1)}P, \quad x^{(n)} = x^{(0)}P^n.$$

In this example, predictions for the weather on more distant days are inaccurate and tend towards a steady state vector. This vector represents the probabilities of sunny and rainy weather on all days, and is independent on the initial weather. The steady state vector is defined as $v = \lim_{n \rightarrow \infty} x^{(n)}$ but only converges if P is a regular transitions matrix (that is there is at least one P^n with all non-zero entries). Since v is independent of initial conditions, it must be unchanged when transformed by P . This makes it an eigenvector with eigenvalue 1, and means it can be derived from P . We have $vP = v$ or

$$v(P - I) = v \left(\begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = v \begin{bmatrix} -0.1 & 0.1 \\ 0.5 & -0.5 \end{bmatrix} = (0, 0).$$

Thus $-0.1v_1 + 0.5v_2 = 0$ and since v is a probability vector, we also have $v_1 + v_2 = 1$. Solving the system with two unknown variables, we get the steady state distribution

$$v = (v_1, v_2) = (0, 833, -167).$$

In conclusion, in the long term, 83% of days are sunny.

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